Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Assignment 10

To solve for the Example class on 6th May. Submit the solution of Problem 1 by Sunday 4th May if you wish feedback on it. Some hints will be given on Friday 2nd May.

The solution of each problem should be no longer than one page!

Starred problems are typically harder. Don't worry if you cannot solve them.

Problem 1. It is easy to prove that any graph with n vertices and maximum degree d contains an independent set of size at least $n/(d+1)$. The goal of this exercise is to show that at the price of decreasing the size of such a set by a constant factor we can guarantee that it has a certain structure.

Let $G = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of G into r pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geq 2ed$, where e is the basis of the natural logarithm. Show that there is an independent set of vertices $W \subseteq V$ that contains a vertex from each V_i.

Problem 2. Show that for $d \geq 2$, every d-regular graph $G = (V, E)$ contains a set U such that for every vertex $v \in V$, the neighbourhood $N(v)$ of v satisfies $1 \leq |N(v) \cap U| \leq 50 \log d$.

Problem 3. Let $G = (V, E)$ be a graph with chromatic number $\chi(G)$.

(a) Let $\{U_1, U_2\}$ be a partition of V and denote by $H_i = G[U_i]$ the subgraph induced by U_i for $i \in \{1, 2\}$. Show that $\chi(H_1) + \chi(H_2) \geq \chi(G)$.

(b)* Suppose $\chi(G) = 1000$ and let $U \subseteq V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that:

$$\mathbb{P}[\chi(H) \leq 400] < \frac{1}{100}.$$